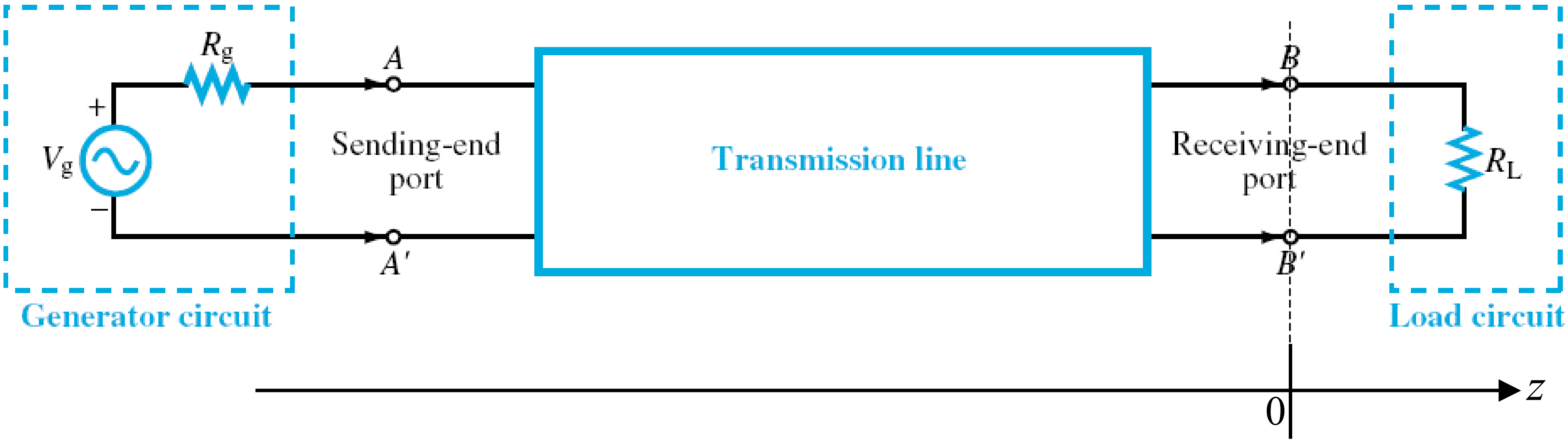
Transmission Lines – Basic Theories

1 Introduction

At high frequencies, the wavelength is much smaller than the circuit size, resulting in different phases at different locations in the circuit.

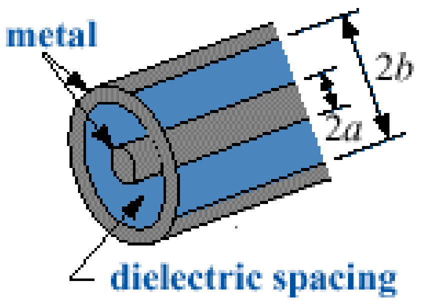
Quasi-static circuit theory cannot be applied. We need to use transmission line theory.



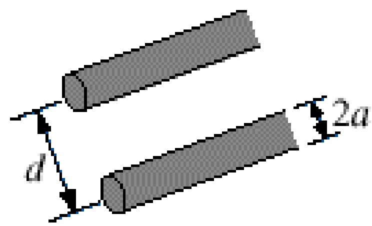
A transmission line is a two-port network connecting a generator circuit at the sending end to a load at the receiving end.

Unlike in circuit theory, the **length** of a transmission line is of utmost importance in transmission line analysis.

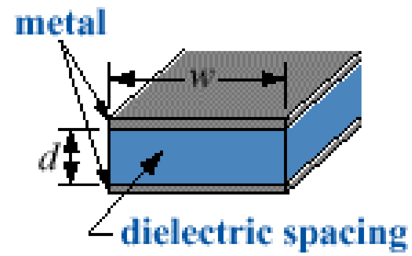
2 Common Types of Transmission Lines



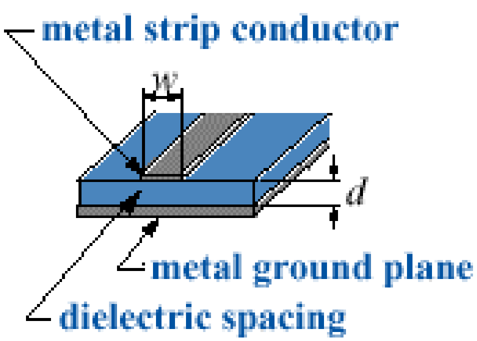
(a) Coaxial line



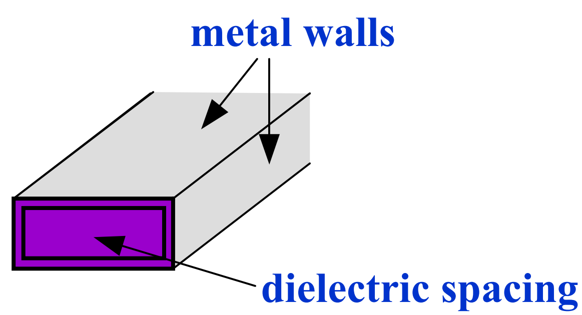
(b) Two-wire line



(c) Parallel-plate line



(d) Microstrip line

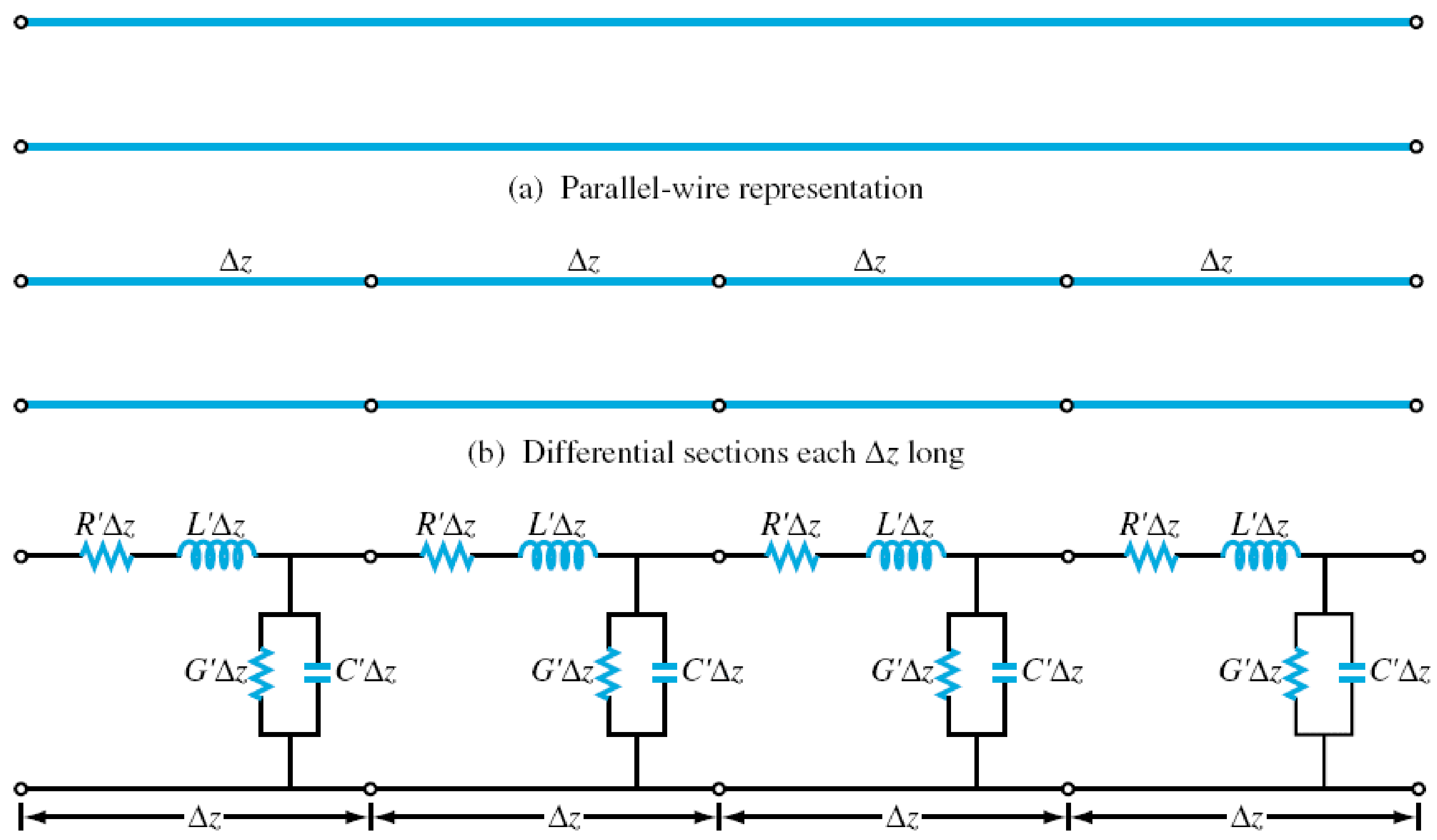


(e) Waveguide

We focus on studying the coaxial and the two-wire transmission lines.

3 AC Steady-State Analysis

3.1 Distributed parameter representation



(c) Each section is represented by an equivalent circuit

We use the following distributed parameters to characterize the circuit properties of a transmission line.

R' = resistance per unit length, **(Ω/m)**
L' = inductance per unit length, **(H/m)**
G' = conductance per unit length, **(S/m)**
C' = capacitance per unit length, **(F/m)**
Δz = increment of length, **(m)**

These parameters are related to the physical properties of the material filling the space between the two wires.

$$L'C' = \mu\varepsilon \qquad \frac{G'}{C'} = \frac{\sigma}{\varepsilon}$$

(See Text Book No.3, pp. 432-433)

where μ, ε, σ = permittivity, permeability, conductivity of the surrounding medium.

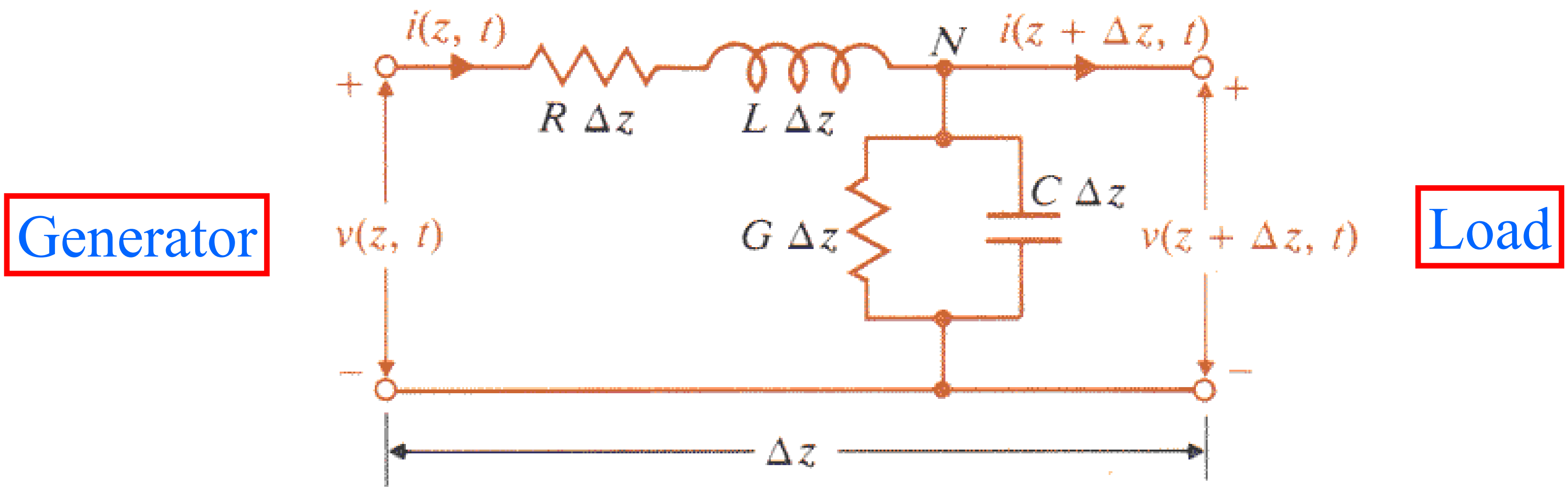
For the coaxial and two-wire transmission lines, the distributed parameters are related to the physical properties and geometrical dimensions as follows:

Parameter	Coaxial	Two Wire
R'	$\frac{R_S}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$	$\frac{R_S}{\pi a}$
L'	$\frac{\mu}{2\pi}\ln(b/a)$	$\frac{\mu}{\pi}\ln\left[(d/2a)+\sqrt{(d/2a)^2-1}\right]$
G'	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\ln\left[(d/2a)+\sqrt{(d/2a)^2-1}\right]}$
C'	$\frac{2\pi\varepsilon}{\ln(b/a)}$	$\frac{\pi\varepsilon}{\ln\left[(d/2a)+\sqrt{(d/2a)^2-1}\right]}$

Surface resistivity of the conductors (See Text Book No.3, pp. 445-447)

3.2 Equations and solutions

Consider a short section Δz of a transmission line (dropping the primes on R', L', G', C' hereafter) :



Using **KVL** and **KCL** circuit theorems, we can derive the following differential equations for this section of transmission line.

$$v\left(z,t\right)-R\Delta z i(z,t)-L\Delta z\frac{\partial i(z,t)}{\partial t}-v\left(z+\Delta z,t\right)=0$$

$$i(z,t)-G\Delta z v(z+\Delta z,t)-C\Delta z\frac{\partial v(z+\Delta z,t)}{\partial t}-i(z+\Delta z,t)=0$$

By letting $\Delta z \rightarrow 0$, these lead to coupled equations:

$$-\frac{\partial v(z,t)}{\partial z}=Ri(z,t)+L\frac{\partial i(z,t)}{\partial t}$$

$$-\frac{\partial i(z,t)}{\partial z}=Gv(z,t)+C\frac{\partial v(z,t)}{\partial t}$$

General Transmission Line Equations – Coupled!

For sinusoidal varying voltages and currents, we can use phasor forms.

$$v(z,t) = \mathrm{Re}\{V(z)e^{j\omega t}\}$$

$$i(z,t) = \mathrm{Re}\{I(z)e^{j\omega t}\}$$

$V(z)$ and $I(z)$ are called **phasors** of $v(z,t)$ and i(z,t). In terms of phasors, the coupled equations can be written as:

$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z)$$

$$-\frac{dI(z)}{dz} = (G + j\omega C)V(z)$$

After decoupling,

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z)$$

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z)$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

γ is the complex propagation constant whose real part α is the **attenuation constant** (Np/m) and whose imaginary part β is the **phase constant** (rad/m). Generally, these quantities are functions of ω.

Solutions to transmission line equations:

Forward travelling wave.

$$V(z) = V^{+}(z) + V^{-}(z)$$
$$= V_0^{+} e^{-\gamma z} + V_0^{-} e^{\gamma z}$$
$$I(z) = I^{+}(z) + I^{-}(z)$$
$$= I_0^{+} e^{-\gamma z} + I_0^{-} e^{\gamma z}$$

Backward travelling wave.

$V_0^{+}, V_0^{-}, I_0^{+}, I_0^{-}$ = wave amplitudes in the forward and backward directions at $z = 0$. (They are complex numbers in general.)

4 Transmission Line Parameters

From the solutions to the transmission line equations, it can be shown (using the coupled transmission line equations) that:

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma}$$

This ratio is called characteristic impedance Z_0.

$$Z_0 = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$\boldsymbol{Z_0}$ and γ are the two most important parameters of a transmission line. They depend on the distributed parameters (*RLGC*) of the line itself and ω but not the length of the line.

Parameters for lossless transmission lines

For lossless transmission lines, $R = G = 0$.

$$\alpha = 0$$

$$\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\varepsilon}$$

$$u_p = \text{phase velocity} = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\gamma = \text{complex propagation constant}$$

$$= j\beta = j\omega\sqrt{\mu\varepsilon} = j2\pi f\sqrt{\mu\varepsilon} = j\frac{2\pi}{\lambda} = jk$$

$$\lambda = \text{wavelength along the transmission line}$$

$$= \frac{u_p}{f} = \frac{1}{f\sqrt{\mu\varepsilon}} = \frac{\omega}{f\beta} = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{LC}}$$

$$Z_0 = \text{characteristic impedance}$$

$$= \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$= \sqrt{\frac{L}{C}}$$

Voltage and current along the line:

$$V(z) = V_0^+ e^{-jkz} + V_0^- e^{jkz}$$

$$I(z) = I_0^+ e^{-jkz} + I_0^- e^{jkz}$$

Define a reflection coefficient at $z = 0$ as Γ_L:

$$\boxed{\begin{aligned}\Gamma_L &= \frac{\text{reflected voltage at } z = 0}{\text{incident voltage at } z = 0} \\ &= \frac{V_0^- e^{jk\times 0}}{V_0^+ e^{-jk\times 0}} = \frac{V_0^-}{V_0^+} = |\Gamma_L| e^{j\theta_L}\end{aligned}}$$

In terms of the reflection coefficient Γ_L, the total voltage and current can be written as:

$$
\begin{aligned}
V(z) &= V_0^+ e^{-jkz} + V_0^- e^{jkz} \\
&= V_0^+ e^{-jkz}\left(1 + \frac{V_0^-}{V_0^+} e^{2jkz}\right) \\
&= V_0^+ e^{-jkz}\left(1 + \Gamma_L e^{2jkz}\right)
\end{aligned}
$$

$$
\begin{aligned}
I(z) &= \frac{V_0^+}{Z_0} e^{-jkz} - \frac{V_0^-}{Z_0} e^{jkz} \\
&= \frac{V_0^+}{Z_0} e^{-jkz}\left(1 - \frac{V_0^-}{V_0^+} e^{2jkz}\right) \\
&= I_0^+ e^{-jkz}\left(1 - \Gamma_L e^{2jkz}\right)
\end{aligned}
$$

In subsequent analyses, we will consider only lossless transmission lines.

5 Infinitely Long Transmission Line

For an infinitely long transmission line, there can be no reflected wave (backward travelling wave). So for an infinite long transmission line, there is only a forward travelling wave.

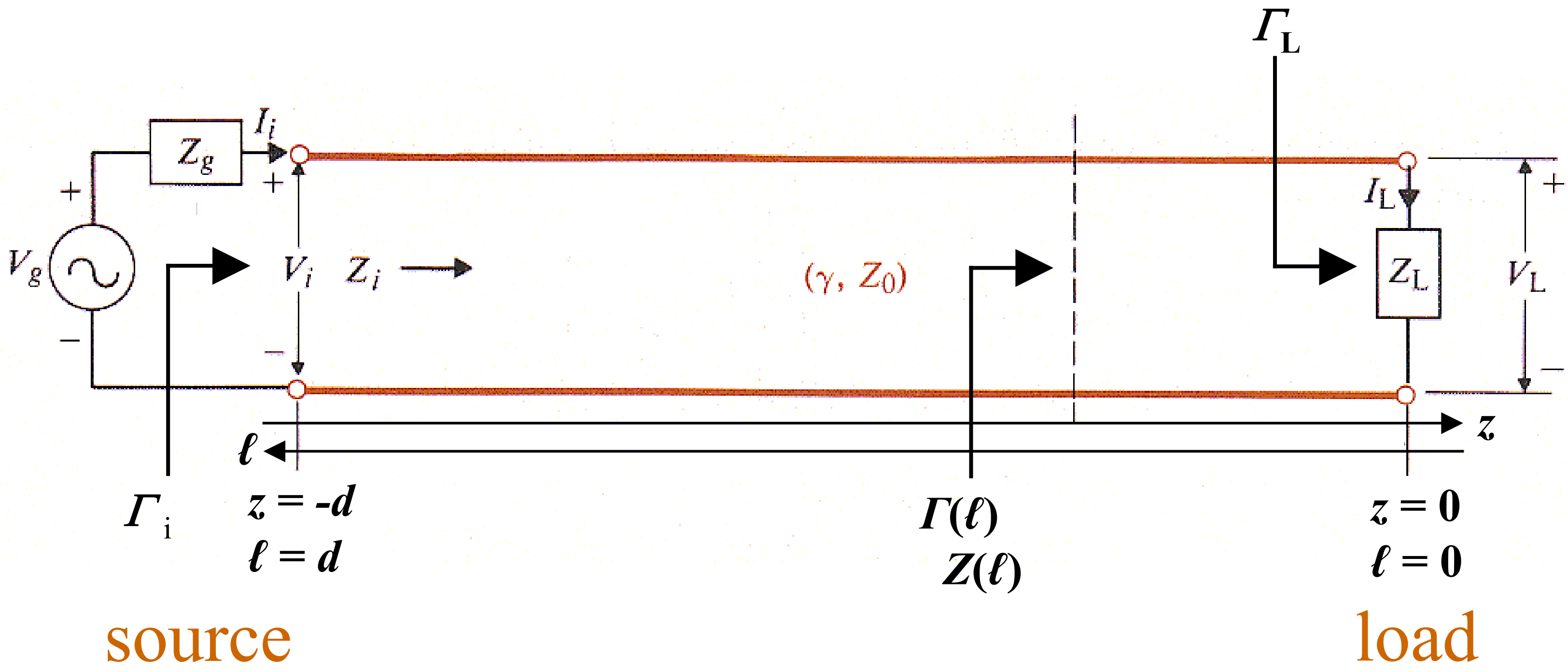
$$V(z) = V^{+}(z) = V_0^{+} e^{-jkz}$$

$$I(z) = I^{+}(z) = I_0^{+} e^{-jkz}$$

$$Z(z) = \frac{V(z)}{I(z)} = \frac{V_0^{+}(z)}{I_0^{+}(z)} = Z_0$$

$$\Gamma_L = 0$$

6 Terminated Transmission Line



Note the two coordinate systems and their relation:

z = measuring from the left to the right

ℓ = measuring from the right to the left

$\boxed{\ell = -z}$

In the z coordinate system,

$$V_0^+ e^{-jkz} + V_0^- e^{jkz} = V(z)$$

$$I_0^+ e^{-jkz} + I_0^- e^{jkz} = I(z)$$

In the ℓ (ℓ = -z) coordinate system,

$$V_0^+ e^{jk\ell} + V_0^- e^{-jk\ell} = V(\ell)$$

$$I_0^+ e^{jk\ell} + I_0^- e^{-jk\ell} = I(\ell)$$

We will use the ℓ coordinate system in subsequent analyses.

The characteristic impedance in the ℓ coordinate system is:

$$\frac{V_0^+}{I_0^+} = Z_0$$

The reflection coefficient at $\ell = 0$ in the ℓ coordinate system is:

$$\Gamma(\ell = 0) = \frac{V_0^- e^{-jk\times 0}}{V_0^+ e^{jk\times 0}} = \Gamma_L$$

As Γ_L is obtained at $\ell = 0$ (the load position), it is called the reflection coefficient at the load.

At the position of the load ($\ell = 0$), the voltage is V_L and the current is I_L. Then we have:

$$V_0^+ + V_0^- = V_L$$

$$\boxed{\frac{V_L}{I_L} = Z_L}$$

$$\frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} = I_L$$

Solve these two equations, we have:

$$V_0^+ = \frac{1}{2} I_L \left(Z_L + Z_0 \right)$$

$$V_0^- = \frac{1}{2} I_L \left(Z_L - Z_0 \right)$$

Putting the expressions for V_0^+ and V_0^- into the equations for the voltage and current, we have:

$$\boxed{\begin{aligned} V(\ell) &= \frac{1}{2} I_L \left[Z_L \left(e^{jk\ell} + e^{-jk\ell} \right) + Z_0 \left(e^{jk\ell} - e^{-jk\ell} \right) \right] \\ &= I_L \left[Z_L \cos(k\ell) + jZ_0 \sin(k\ell) \right] \end{aligned}}$$

$$\boxed{\begin{aligned} I(\ell) &= \frac{1}{2} \frac{I_L}{Z_0} \left[Z_L \left(e^{jk\ell} - e^{-jk\ell} \right) + Z_0 \left(e^{jk\ell} + e^{-jk\ell} \right) \right] \\ &= \frac{I_L}{Z_0} \left[Z_0 \cos(k\ell) + jZ_L \sin(k\ell) \right] \end{aligned}}$$

Using $V(\ell)$ and $I(\ell)$, we can obtain the impedance $Z(\ell)$ at an arbitrary point ℓ on the transmission line as:

$$\boxed{Z(\ell) = \frac{V(\ell)}{I(\ell)} = Z_0 \frac{Z_L + jZ_0 \tan(k\ell)}{Z_0 + jZ_L \tan(k\ell)}}$$

The reflection coefficient at the load Γ_L can be expressed as:

$$\boxed{\Gamma_L = \frac{V_0^-}{V_0^+} = \frac{\frac{1}{2} I_L (Z_L - Z_0)}{\frac{1}{2} I_L (Z_L + Z_0)} = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

In fact, we can further define a reflection coefficient $\Gamma(\ell)$ at any point ℓ on the transmission line by:

$$\Gamma(\ell) = \frac{\text{reflected voltage at point } \ell}{\text{incident voltage at point } \ell}$$

$$= \frac{V_0^- e^{-jk\ell}}{V_0^+ e^{jk\ell}} = \frac{V_0^-}{V_0^+} e^{-j2k\ell} = \Gamma_L e^{-j2k\ell}$$

As we know (by solving the two equations on page 22 with $\ell \neq 0$):

$$V_0^+ e^{jk\ell} = \frac{1}{2} I(\ell)\big(Z(\ell) + Z_0\big)$$

$$V_0^- e^{-jk\ell} = \frac{1}{2} I(\ell)\big(Z(\ell) - Z_0\big)$$

Therefore, alternatively we can write,

$$\boxed{\Gamma(\ell)=\frac{\frac{1}{2}I(\ell)\left[Z(\ell)-Z_0\right]}{\frac{1}{2}I(\ell)\left[Z(\ell)+Z_0\right]}=\frac{Z(\ell)-Z_0}{Z(\ell)+Z_0}}$$

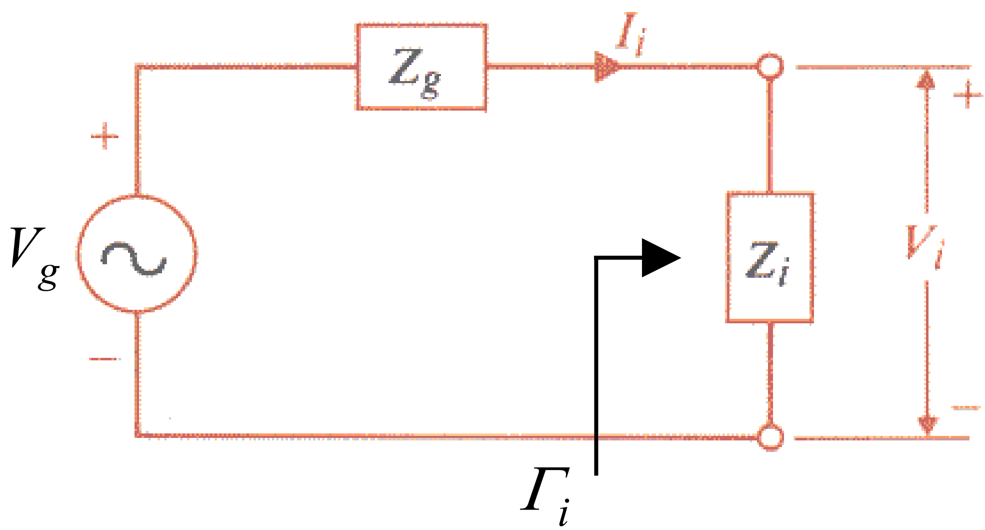
Then,

$$\boxed{Z(\ell)=Z_0\frac{1+\Gamma(\ell)}{1-\Gamma(\ell)}}$$

At the position of the generator ($\ell = d$),

$$\boxed{Z_i = Z(\ell = d) = Z_0 \frac{Z_L + jZ_0 \tan(kd)}{Z_0 + jZ_L \tan(kd)}}$$

$$\boxed{\Gamma(\ell = d) = \Gamma_i = \frac{Z_i - Z_0}{Z_i + Z_0} = \Gamma_L e^{-j2kd}}$$

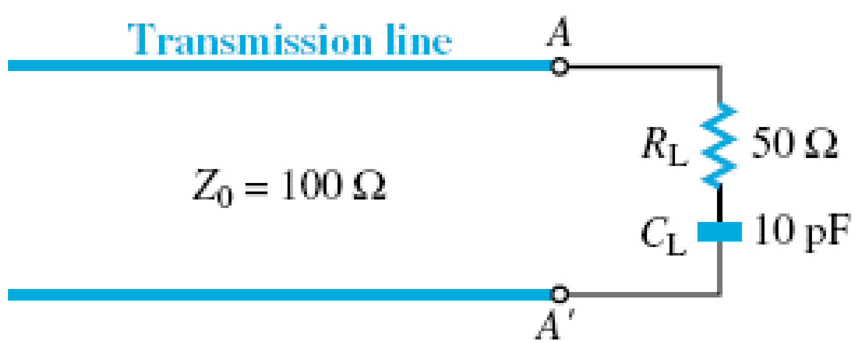


Example 1

A 100-Ω transmission line is connected to a load consisted of a 50-Ω resistor in series with a 10-pF capacitor.

(a) Find the reflection coefficient Γ_L at the load for a 100-MHz signal.

(b) Find the impedance Z_{in} at the input end of the transmission line if its length is 0.125λ.



Solutions

The following information is given

$$R_{\mathrm{L}} = 50\Omega, \quad C_{\mathrm{L}} = 10^{-11}\,\mathrm{F}, \quad Z_0 = 100\Omega, \quad f = 100\,\mathrm{MHz} = 10^8\,\mathrm{Hz}$$

The load impedance is

$$Z_{\mathrm{L}} = R_{\mathrm{L}} - j/\omega C_{\mathrm{L}}$$

$$= 50 - j\frac{1}{2\pi \times 10^8 \times 10^{-11}} = 50 - j159 \quad (\Omega)$$

(a) Voltage reflection coefficient is

$$\Gamma_L = \frac{Z_{\mathrm{L}}/Z_0 - 1}{Z_{\mathrm{L}}/Z_0 + 1} = \frac{0.5 - j1.59 - 1}{0.5 - j1.59 + 1} = 0.76\angle -60.70^\circ$$

(b) $d = 0.125\,\lambda$

$$Z_{in} = Z(\ell = 0.125\lambda)$$

$$= Z_0 \frac{Z_L + jZ_0 \tan(\pi/4)}{Z_0 + jZ_L \tan(\pi/4)}$$

$$= Z_0 \frac{Z_L + jZ_0}{Z_0 + jZ_L}$$

$$= 14.3717 - j25.5544 \quad (\Omega)$$

$$= 29.32\angle -60.65^\circ \quad (\Omega)$$

Normalized $z_{in} = 0.1437 - j\,0.2555\ \Omega$

See animation "Transmission Line Impedance Calculation"

6.1 Voltage/current maxima and minima

$$V(\ell) = V_0^+ e^{jk\ell} + V_0^- e^{-jk\ell}$$

$$= V_0^+ e^{jk\ell}\left(1 + \frac{V_0^-}{V_0^+} e^{-j2k\ell}\right)$$

$$= V_0^+ e^{jk\ell}\left(1 + \Gamma_L e^{-j2k\ell}\right)$$

$$\Gamma_L = \left|\Gamma_L\right| e^{j\theta_L}$$

$$|\Gamma_L| \le 1$$

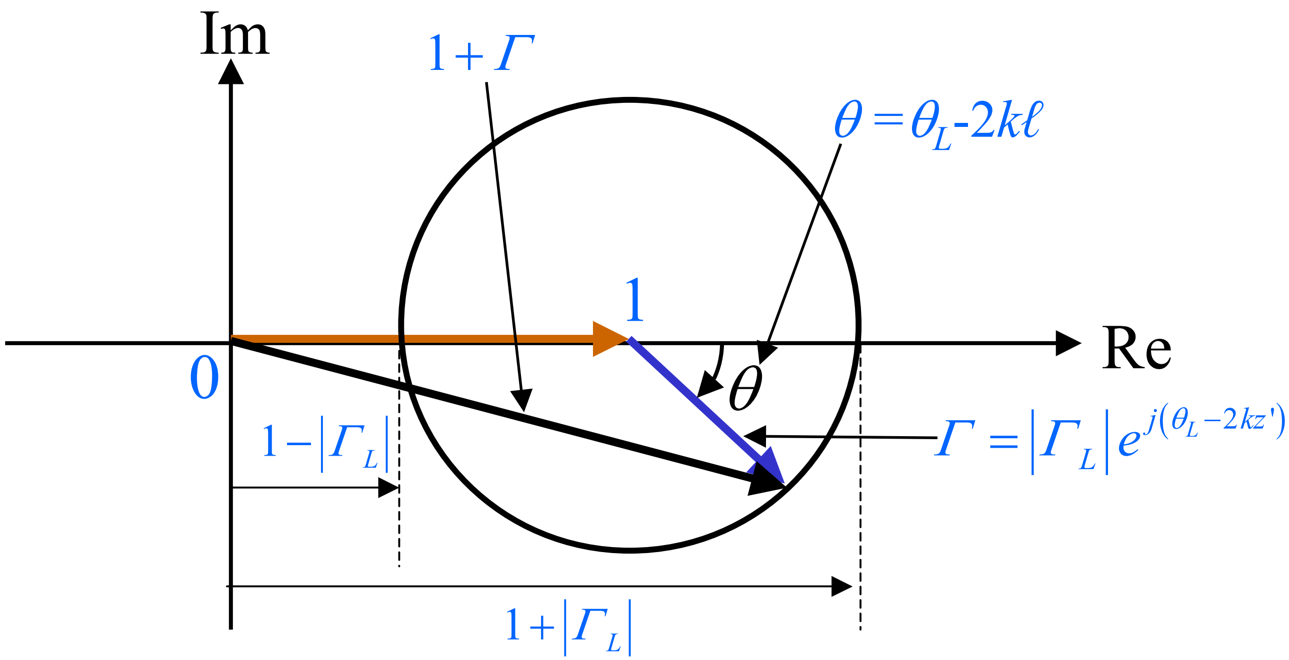
$$\left|V(\ell)\right| = \left|V_0^+\right|\left|1 + \Gamma_L e^{-j2k\ell}\right|$$

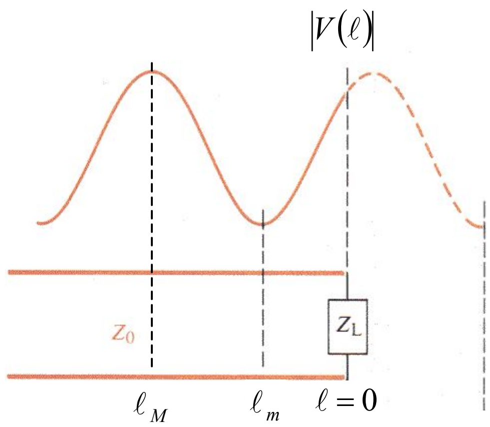
$$= \left|V_0^+\right|\left|1 + \left|\Gamma_L\right| e^{j(\theta_L - 2k\ell)}\right|$$

$$= \left|V_0^+\right|\left|1 + \Gamma\right|$$

$$\Gamma = \left|\Gamma_L\right| e^{j(\theta_L - 2k\ell)}$$

= a complex number





Complex plane of $(1+\Gamma)$

See animation "Transmission Line Voltage Maxima and Minima"

$\left|V(\ell)\right|$ is maximum when $\left|1+\Gamma\right| = \left(1+\left|\Gamma_L\right|\right)$

$$\left|V(\ell)\right|_{\max} \Rightarrow \quad \theta = \theta_L - 2k\ell = -2n\pi$$

$$\Rightarrow \quad \ell_M = \frac{\theta_L \lambda}{4\pi} + \frac{n\lambda}{2}, \quad n = 0, 1, 2, \cdots$$

Note: θ_L has to be specified in the range $[-\pi, \pi)$.

$\left|V(\ell)\right|$ is minimum when $\left|1-\Gamma\right| = \left(1-\left|\Gamma_L\right|\right)$

$$\left|V(\ell)\right|_{\min} \Rightarrow \quad \theta = \theta_L - 2k\ell = -(2n+1)\pi$$

$$\Rightarrow \quad \ell_m = \frac{\theta_L \lambda}{4\pi} + \frac{(2n+1)\lambda}{4}, \quad n = 0, 1, 2, \cdots$$

Note: θ_L has to be specified in the range $[-\pi, \pi)$.

As current is

$$\left|I(\ell)\right| = \left|I_0^+\right|\left|1 - \Gamma_L e^{-j2k\ell}\right|$$

$$= \left|\frac{V_0^+}{Z_0}\right|\left|1 - \Gamma\right|$$

Current is maximum when voltage is minimum and minimum when voltage is maximum.

$$\boxed{\left|I(\ell)\right|_{\max} \text{ at } \ell_M = \frac{\theta_L \lambda}{4\pi} + \frac{(2n+1)\lambda}{4}, \; n = 0,1,2,\cdots, \text{ with } \left|\theta_L\right| \le \pi}$$

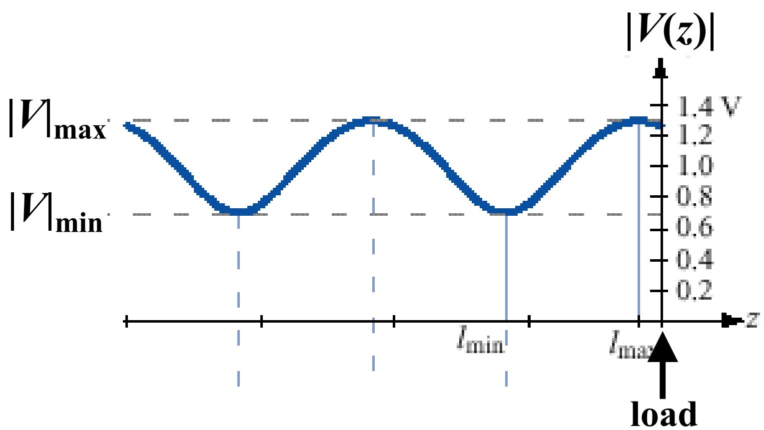
$$\boxed{\left|I(\ell)\right|_{\min} \text{ at } \ell_m = \frac{\theta_L \lambda}{4\pi} + \frac{n\lambda}{2}, \; n = 0,1,2,\cdots, \text{ with } \left|\theta_L\right| \le \pi}$$

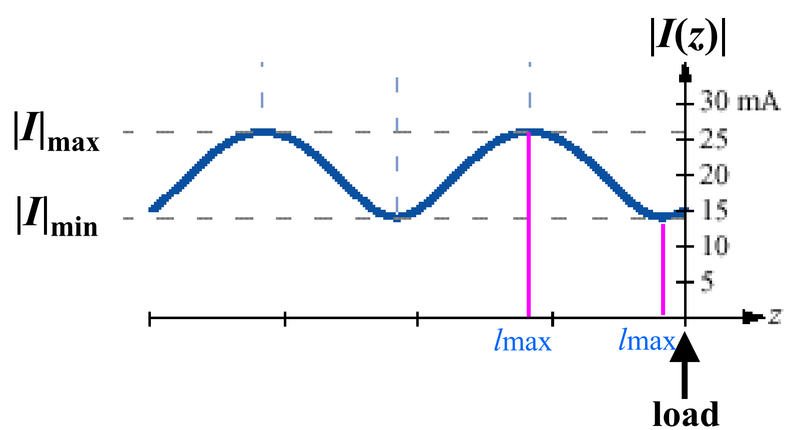
Define a voltage standing wave ratio (VSWR) as:

$$S = \text{voltage standing wave ratio (VSWR)}$$

$$= \frac{\left|V(\ell)\right|_{\max}}{\left|V(\ell)\right|_{\min}} = \frac{\left|V_0^+\right|\left(1+\left|\Gamma_L\right|\right)}{\left|V_0^+\right|\left(1-\left|\Gamma_L\right|\right)} = \frac{1+\left|\Gamma_L\right|}{1-\left|\Gamma_L\right|} \qquad \text{(dimensionless)}$$

$$\boxed{\left|\Gamma_L\right| = \frac{S-1}{S+1}}$$





Special terminations

Γ_L	S	Z_L
0	1	$Z_L = Z_0$ (matched)
-1	∞	$Z_L = 0$ (short-circuited)
1	∞	$Z_L = \infty$ (open-circuited)

6.2 Power flow in a transmission line

Power flow at any point z on a transmission line is given by:

$$\boxed{P_{av}(z) = \frac{1}{2}\mathrm{Re}\left\{V(z)I^*(z)\right\}}$$

Power delivered by the source:

$$P_s = \frac{1}{2}\mathrm{Re}\left\{V_g I_i^*\right\}$$

Power dissipated in the source impedance Z_g:

$$P_{Z_g} = \frac{1}{2}\mathrm{Re}\left\{V_{Z_g} I_{Z_g}^*\right\} = \frac{1}{2}\mathrm{Re}\left\{Z_g I_i I_i^*\right\} = \frac{1}{2}\left|I_i\right|^2 \mathrm{Re}\left\{Z_g\right\}$$

Power input to the transmission line:

$$P_i = P_{av}(-d) = \frac{1}{2}\mathrm{Re}\{V(-d)I^*(-d)\}$$

$$= \frac{1}{2}\mathrm{Re}\{V_i I_i^*\} = \frac{1}{2}\mathrm{Re}\{Z_i I_i I_i^*\} = \frac{1}{2}|I_i|^2\,\mathrm{Re}\{Z_i\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{V_i \frac{V_i^*}{Z_i^*}\right\} = \frac{1}{2}|V_i|^2\,\mathrm{Re}\left\{\frac{1}{Z_i^*}\right\}$$

Power dissipated in the terminal impedance:

$$P_L = P_{av}(0) = \frac{1}{2}\mathrm{Re}\{V(0)I^*(0)\}$$

$$= \frac{1}{2}\mathrm{Re}\{V_L I_L^*\} = \frac{1}{2}\mathrm{Re}\{Z_L I_L I_L^*\} = \frac{1}{2}|I_L|^2\,\mathrm{Re}\{Z_L\}$$

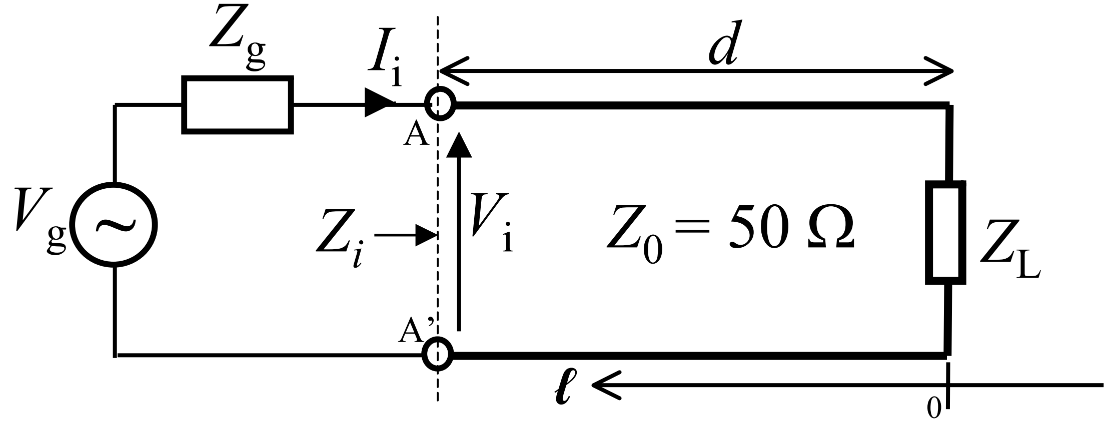
$$= \frac{1}{2}\mathrm{Re}\left\{V_L \frac{V_L^*}{Z_L^*}\right\} = \frac{1}{2}|V_L|^2\,\mathrm{Re}\left\{\frac{1}{Z_L^*}\right\}$$

By the principle of conservation of power:

$$\boxed{\begin{aligned} P_s &= P_{Z_g} + P_i \\ P_i &= P_L \end{aligned}}$$

Example 2

A lossless transmission line with Z_0 = 50 Ω and d =1.5 m connects a voltage V_g source to a terminal load of Z_L = (50 + j50) Ω. If V_g = 60 V, operating frequency f = 100 MHz, and Z_g = 50 Ω, find the distance of the first voltage maximum ℓ_M from the load. What is the power delivered to the load P_L? Assume the speed of the wave along the transmission line equal to speed of light, c.



Solutions

The following information is given:

$$Z_0 = 50\Omega, \quad d = 1.5\,\text{m},$$

$$V_g = 60\,\text{V},\ Z_g = 50\Omega,\ Z_L = 50 + j50\Omega,$$

$$f = 100\text{MHz} = 10^8\,\text{Hz}$$

$$u_p = c \quad \Rightarrow \quad \lambda = \frac{c}{10^8} = 3\,\text{m}$$

The reflection coefficient at the load is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 + j50 - 50}{50 + j50 + 50} = 0.2 + j0.4 = 0.45e^{j1.11}$$

Therefore, $\left|\Gamma_L\right| = 0.45, \quad \theta_L = 1.11\,\text{rad}$

Then, $$\ell_M = \frac{\theta_L \lambda}{4\pi} + \frac{n\lambda}{2}, \quad \text{when } n = 0$$
$$= \frac{1.11\lambda}{4\pi} = 0.09\lambda = 0.27\,\text{m} \;\; \text{(from the load)}$$

The input impedance Z_i looking at the input to the transmission line is:

$$Z_i = Z_0 \frac{Z_L + jZ_0 \tan(kd)}{Z_0 + jZ_L \tan(kd)}$$

$$Z_i = 50\frac{50 + j50 + j50\tan\left(\frac{2\pi}{3}\times 1.5\right)}{50 + j(50 + j50)\tan\left(\frac{2\pi}{3}\times 1.5\right)} = 50 + j50\Omega$$

The current at the input to the transmission line is :

$$I_i = \frac{V_g}{Z_g + Z_i} = \frac{60}{50 + 50 + j50} = 0.48 - j0.24 \text{ A}$$

As the transmission line is lossless, power delivered to the load P_L is equal to the power input to the transmission line P_i. Hence,

$$P_L = P_i = \frac{1}{2}|I_i|^2 \operatorname{Re}\{Z_i\} = \frac{1}{2}\times 0.288\times 50 = 7.2 \text{ W}$$

6.3 Complete solutions for voltage and current

The voltage and current on the transmission line can be written as:

$$V(\ell) = V_0^+ e^{jk\ell} + V_0^- e^{-jk\ell} = V_0^+ e^{jk\ell}\left(1 + \Gamma_L e^{-j2k\ell}\right)$$

$$I(\ell) = \frac{V_0^+}{Z_0} e^{jk\ell} - \frac{V_0^+}{Z_0} e^{-jk\ell} = \frac{V_0^+}{Z_0} e^{jk\ell}\left(1 - \Gamma_L e^{-j2k\ell}\right)$$

We still have one unknown V_0^+ in $V(\ell)$ and $I(\ell)$. We need the knowledge of voltage source V_g to further determine V_0^+.

At $\ell = d$, $V(d) = V_i$ and $I(d) = I_i$.

$$V_i = V_0^+ e^{jkd}\left(1 + \Gamma_L e^{-j2kd}\right) \qquad I_i = \frac{V_0^+}{Z_0} e^{jkd}\left(1 - \Gamma_L e^{-j2kd}\right)$$

V_i and I_i are related to the source voltage V_g as:

$$V_g = V_i + I_i Z_g$$

From the expressions of V_i, I_i, and V_g, we can find V_0^+.

$$V_0^+ = \frac{V_g Z_0 e^{-jkd}}{\left(Z_g + Z_0\right)\left(1 - \Gamma_g \Gamma_L e^{-j2kd}\right)}$$

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} = \text{source reflection coefficient}$$

Putting V_0^+ into the expressions of $V(\ell)$ and $I(\ell)$, we have:

$$\boxed{\begin{aligned} V(\ell) &= \frac{V_g Z_0 e^{-jkd}}{\left(Z_g + Z_0\right)\left(1 - \Gamma_g \Gamma_L e^{-j2kd}\right)} e^{jk\ell}\left(1 + \Gamma_L e^{-j2k\ell}\right) \\ I(\ell) &= \frac{V_g e^{-jkd}}{\left(Z_g + Z_0\right)\left(1 - \Gamma_g \Gamma_L e^{-j2kd}\right)} e^{jk\ell}\left(1 - \Gamma_L e^{-j2k\ell}\right) \end{aligned}}$$

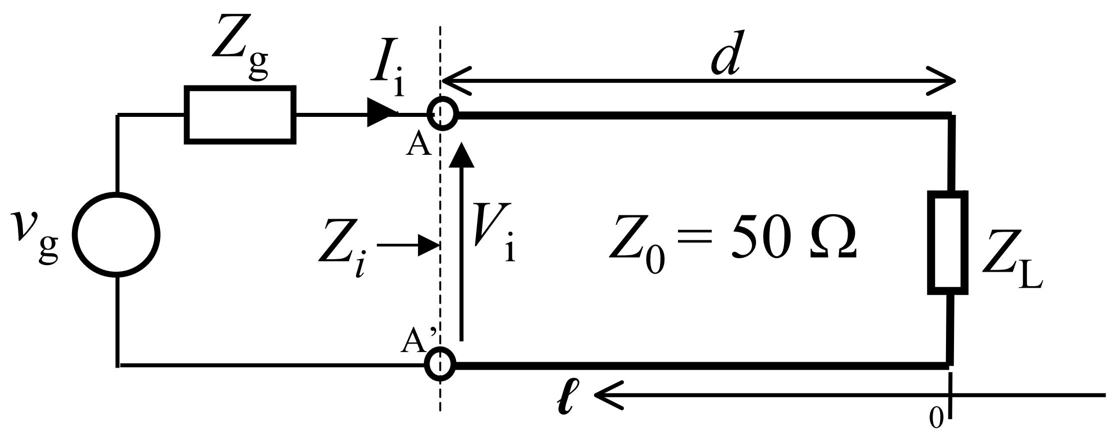
Now the voltage and current on the transmission line are expressed in terms of the known parameters of the transmission line.

Example 3

A 1.05-GHz generator circuit with a series impedance $Z_g = 10\Omega$ and voltage source given by:

$$v_g(t) = 10\sin(\omega t + 30°)\ \ (\mathrm{V})$$

is connected to a load $Z_L = (100 + j50)$ through a 50-Ω, 67-cm-long lossless transmission line. The phase velocity of the line is $0.7c$, where c is the velocity of light in a vacuum. Find the instantaneous voltage and current $v(\ell,t)$ and $i(\ell,t)$ on the line and the average power delivered to the load.



Solutions

$$\lambda = \frac{u_{\mathrm{p}}}{f} = \frac{0.7 \times 3 \times 10^8}{1.05 \times 10^9} = 0.2\ \mathrm{m}$$

$$d = 67\mathrm{cm} = \frac{0.67}{0.2} = 3.35\lambda$$

source reflection coefficient Γ_g

$$= \frac{Z_g - Z_0}{Z_g + Z_0} = \frac{10 - 50}{10 + 50} = -\frac{2}{3}$$

load reflection coefficient Γ_L

$$= \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 + j50 - 50}{100 + j50 + 50} = 0.45 e^{j0.46}$$

$$v_{\mathrm{g}}(t) = 10\sin(\omega t + 30^\circ)$$
$$= 10\cos(\omega t - 60^\circ) = \mathrm{Re}\left\{10e^{-j60^\circ}e^{j\omega t}\right\} \quad (\mathrm{V})$$

Phasor form: $\quad V_{\mathrm{g}} = 10e^{-j\pi/3}$

$$V(\ell) = \frac{V_g Z_0 e^{-jkd}}{(Z_g + Z_0)(1 - \Gamma_g \Gamma_L e^{-j2kd})} e^{jk\ell}\left(1 + \Gamma_L e^{-j2k\ell}\right)$$

$$= \frac{10e^{-j\pi/3}\,50e^{-j\frac{2\pi}{\lambda}(3.35\lambda)}}{(10+50)\left[1 - (-2/3)\left(0.45e^{j0.46}\right)e^{-j\frac{4\pi}{\lambda}(3.35\lambda)}\right]} \times$$
$$\left[e^{jk\ell} + \left(0.45e^{j0.46}\right)e^{-jk\ell}\right] = 10.18e^{j2.77}\left[e^{jk\ell} + 0.45e^{-j(k\ell - 0.46)}\right]$$

$$I(\ell)=\frac{V_g e^{-jkd}}{(Z_g+Z_0)(1-\Gamma_g\Gamma_L e^{-j2kd})}e^{jk\ell}(1-\Gamma_L e^{-j2k\ell})$$
$$=0.20e^{j2.77}\left[e^{jk\ell}-0.45e^{-j(k\ell-0.46)}\right]$$

Therefore instantaneous forms are:

$$v(\ell,t)=\mathrm{Re}\{V(\ell)e^{j\omega t}\}$$
$$=\mathrm{Re}\{10.18e^{j2.77}\left[e^{jk\ell}+0.45e^{-j(k\ell-0.46)}\right]e^{j\omega t}\}$$
$$=10.18\cos(\omega t+k\ell+2.77)+4.58\cos(\omega t-k\ell+3.23)$$
$$i(\ell,t)=\mathrm{Re}\{I(\ell)e^{j\omega t}\}$$
$$=\mathrm{Re}\{0.20e^{j2.77}\left[e^{jk\ell}-0.45e^{-j(k\ell-0.46)}\right]e^{j\omega t}\}$$
$$=0.20\cos(\omega t+k\ell+2.77)-0.09\cos(\omega t-k\ell+3.23)$$

$$Z_i = Z_0 \frac{Z_L + jZ_0 \tan(kd)}{Z_0 + jZ_L \tan(kd)}$$

$$= 50 \frac{(100 + j50) + j50 \tan\left(\frac{2\pi}{\lambda} \times 3.35\lambda\right)}{50 + j(100 + j50)\tan\left(\frac{2\pi}{\lambda} \times 3.35\lambda\right)}$$

$$= 21.9 + j17.4\,\Omega$$

$$I_i = \frac{V_g}{Z_g + Z_i} = \frac{10e^{-j\pi/3}}{10 + 21.9 + j17.4} = 0.28e^{-j1.55}$$

Power delivered to the load

= power input to the transmission line at AA’

$$=\frac{1}{2}\mathrm{Re}\left\{V_i I_i^*\right\}$$

$$=\frac{1}{2}\mathrm{Re}\left\{I_i Z_i I_i^*\right\}$$

$$=\frac{1}{2}\left|I_i\right|^2\mathrm{Re}\left\{Z_i\right\}$$

$$=\frac{1}{2}0.28^2\,\mathrm{Re}\left\{21.9+j17.4\right\}$$

$$=0.86\ \mathrm{Watt}$$

7 Special Cases of Terminations in a Transmission Line

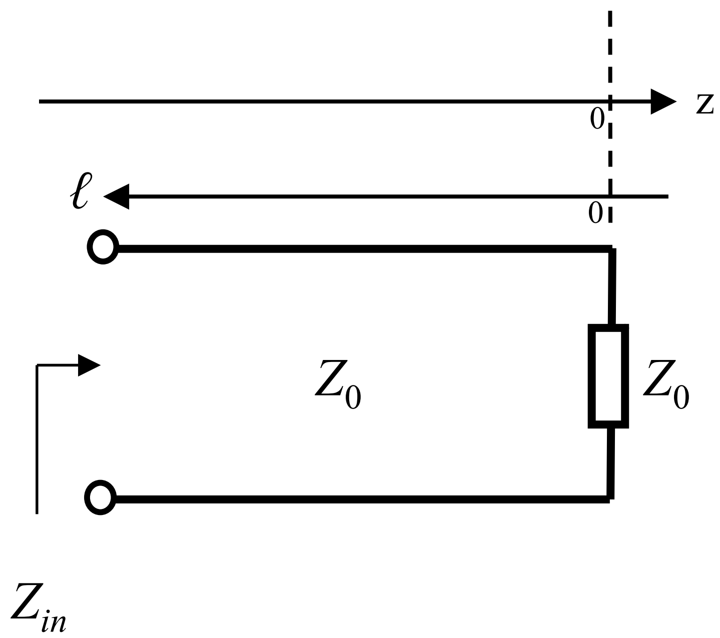
7.1 Matched line

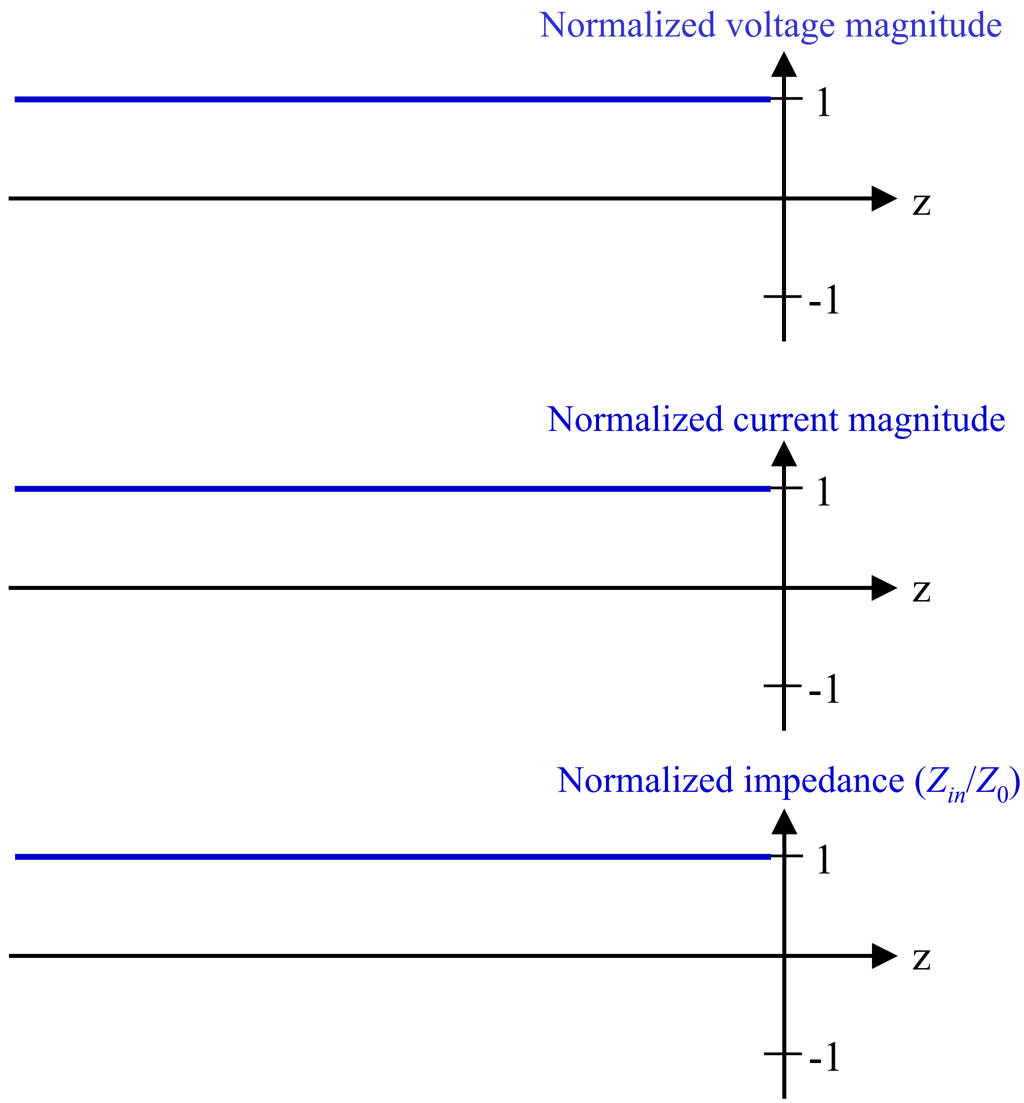
For a matched line, $Z_L = Z_0$. Then,

$$\left.\begin{aligned} Z(\ell) &= Z_0 \frac{Z_0 + jZ_0 \tan(k\ell)}{Z_0 + jZ_0 \tan(k\ell)} = Z_0 \\ \Gamma(\ell) &= \frac{Z(\ell) - Z_0}{Z(\ell) + Z_0} = 0 \end{aligned}\right\} \text{for any length } \ell \text{ of the line}$$

$\boxed{\text{Note } \ell = -z}$

Thus, there is no reflection on a matched line. There is only an incident voltage. It is same as the case of an infinitely long line.



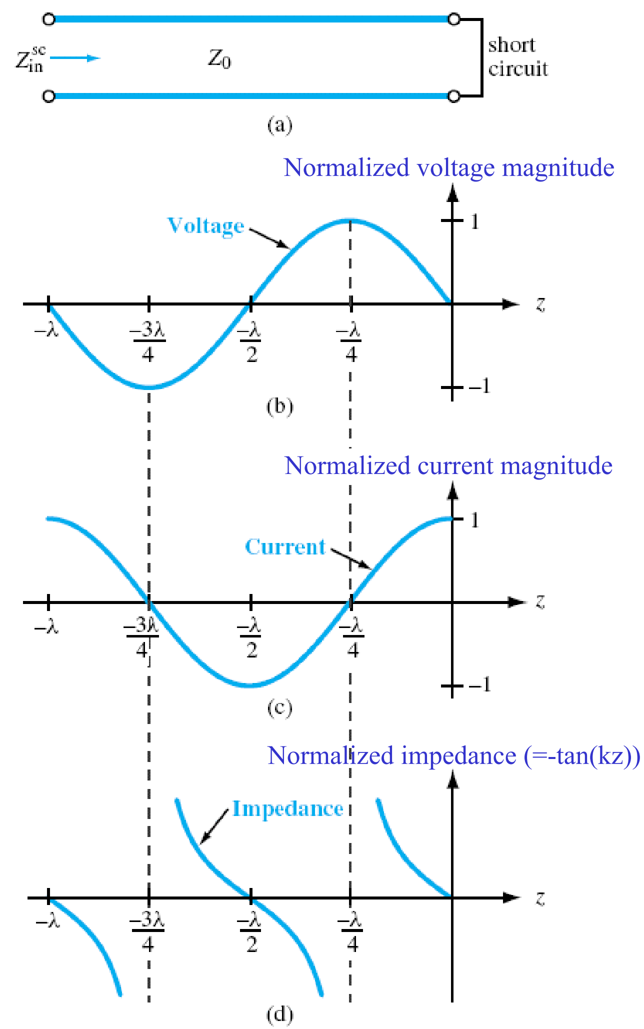


Note:
Normalized voltage = voltage/max. |voltage|
Normalized current = current/max. |current|

7.2 Short-circuited line

For a short circuit, $Z_L = 0$. Then

$$Z_{\text{in}}^{\text{sc}} = jZ_0 \tan(k\ell) = -jZ_0 \tan(kz)$$



7.3 Open-circuited line

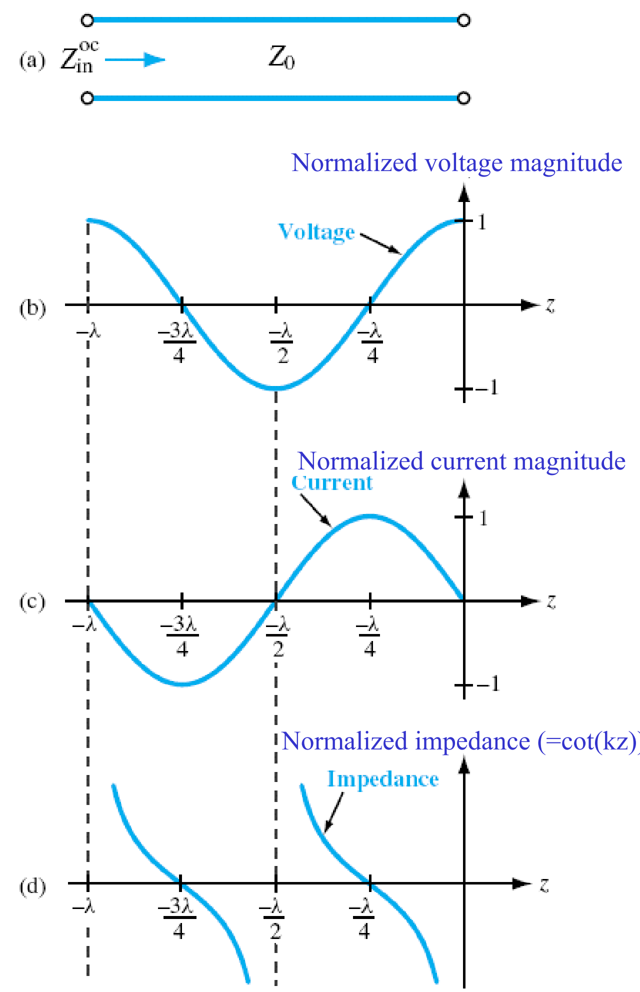
For an open circuit, $Z_L = \infty$. Then

$$Z_{\text{in}}^{\text{oc}} = -jZ_0 \cot(k\ell) = jZ_0 \cot(kz)$$

Note that:

$$Z_{\text{in}}^{\text{sc}} Z_{\text{in}}^{\text{oc}} = [jZ_0 \tan(k\ell)][-jZ_0 \cot(k\ell)]$$
$$= Z_0^2$$

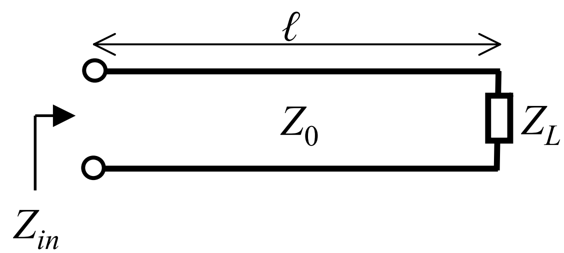
$$Z_{\text{in}}^{\text{sc}} / Z_{\text{in}}^{\text{oc}} = [jZ_0 \tan(k\ell)]/[-jZ_0 \cot(k\ell)]$$
$$= -\tan^2(k\ell)$$



Given $Z_{\text{in}}^{\text{sc}}$, $Z_{\text{in}}^{\text{oc}}$, and ℓ, compute Z_0 and k.

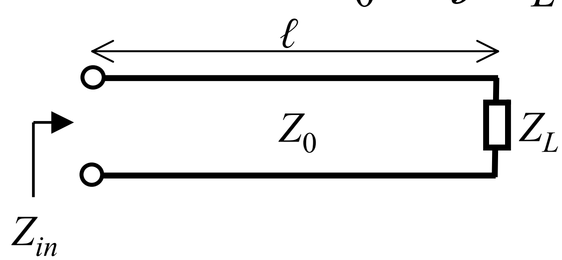
7.4 λ/4 transmission line terminated in Z_L

$$Z_{in} = Z(\ell = \lambda/4) = Z_0 \frac{Z_L + jZ_0 \tan(\pi/2)}{Z_0 + jZ_L \tan(\pi/2)} = \frac{Z_0^2}{Z_L}$$



7.5 λ/2 transmission line terminated in Z_L

$$Z_{in} = Z(\ell = \lambda/2) = Z_0 \frac{Z_L + jZ_0 \tan(\pi)}{Z_0 + jZ_L \tan(\pi)} = Z_L$$



Example 4

The open-circuit and short-circuit impedances measured at the input terminals of a lossless transmission line of length 1.5 m (which is less than a quarter wavelength) are $-j54.6\ \Omega$ and $j103\ \Omega$, respectively.

(a) Find Z_0 and k of the line.

(b) Without changing the operating frequency, find the input impedance of a short-circuited line that is twice the given length.

(c) How long should the short-circuited line be in order for it to appear as an open circuit at the input terminals?

Solution

The given quantities are

$$Z_{\text{in}}^{\text{oc}} = -j54.6\,\Omega$$

$$Z_{\text{in}}^{\text{sc}} = j103\,\Omega$$

$$\ell = 1.5\text{m}$$

(a) $Z_0 = \sqrt{Z_{\text{in}}^{\text{oc}} Z_{\text{in}}^{\text{sc}}} = 75\,\Omega$

$$k = \frac{1}{\ell}\tan^{-1}\sqrt{-Z_{\text{in}}^{\text{sc}} / Z_{\text{in}}^{\text{oc}}} = 0.628 \quad \text{rad/m}$$

$$\lambda = \frac{2\pi}{k} = 10\text{m}$$

(b) For a line twice as long, $\ell = 3$ m and $k\,\ell = 1.884$ rad,

$$Z_{\text{in}}^{\text{sc}} = jZ_0 \tan k\ell = -j232\,\Omega$$

(c) Short circuit input impedance

$$= Z_{\text{in}}^{\text{sc}} = jZ_0 \tan(k\ell)$$

$$\text{For } Z_{\text{in}}^{\text{sc}} = \infty, \quad \Rightarrow \; k\ell = \pi/2 + n\pi, \quad n = 0,1,2,\cdots$$

$$\ell = \frac{\pi/2 + n\pi}{k} = \frac{2n+1}{4}\lambda$$